EGC 2018

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Long-Range Influences in (Social) Networks

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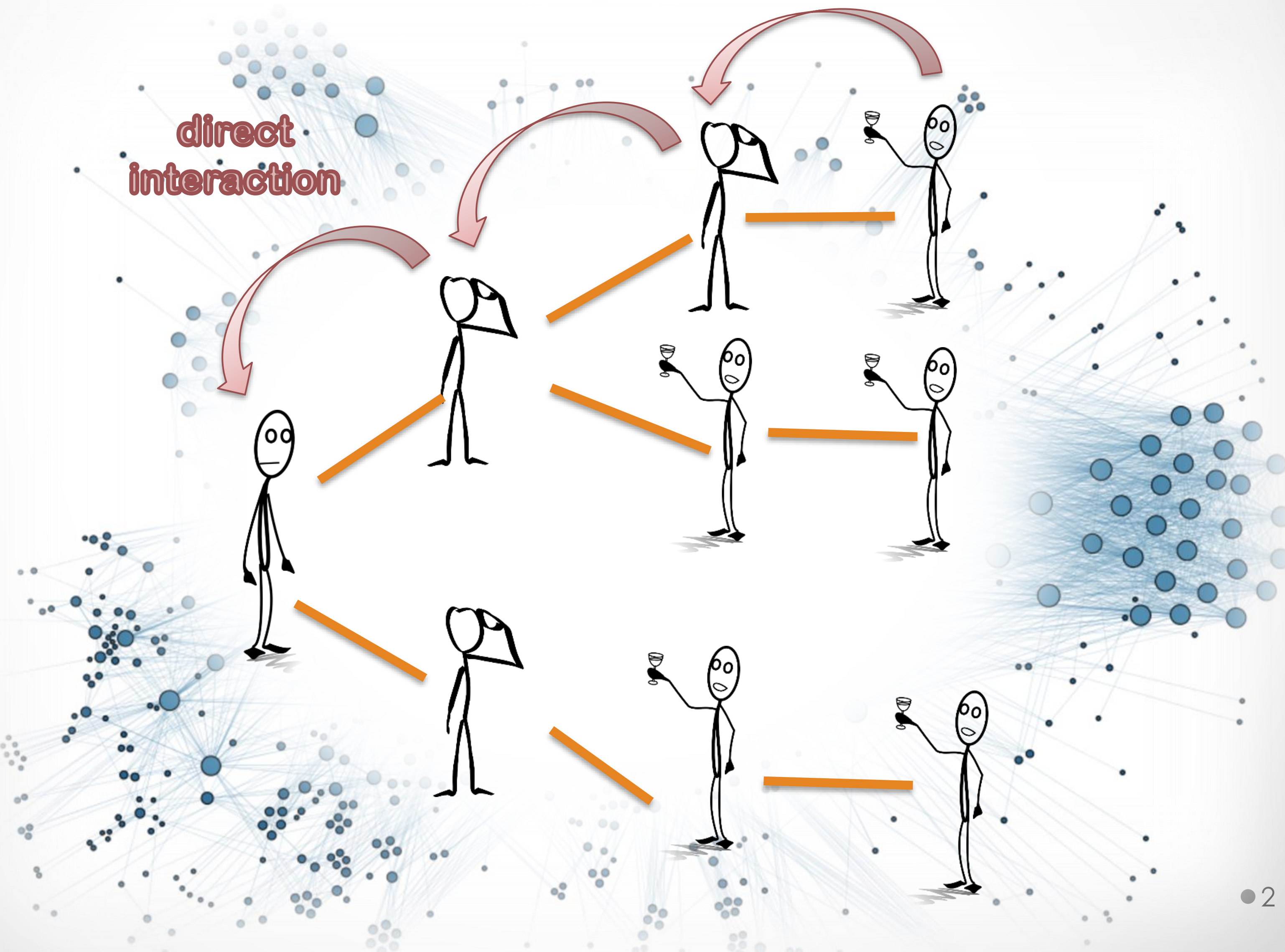
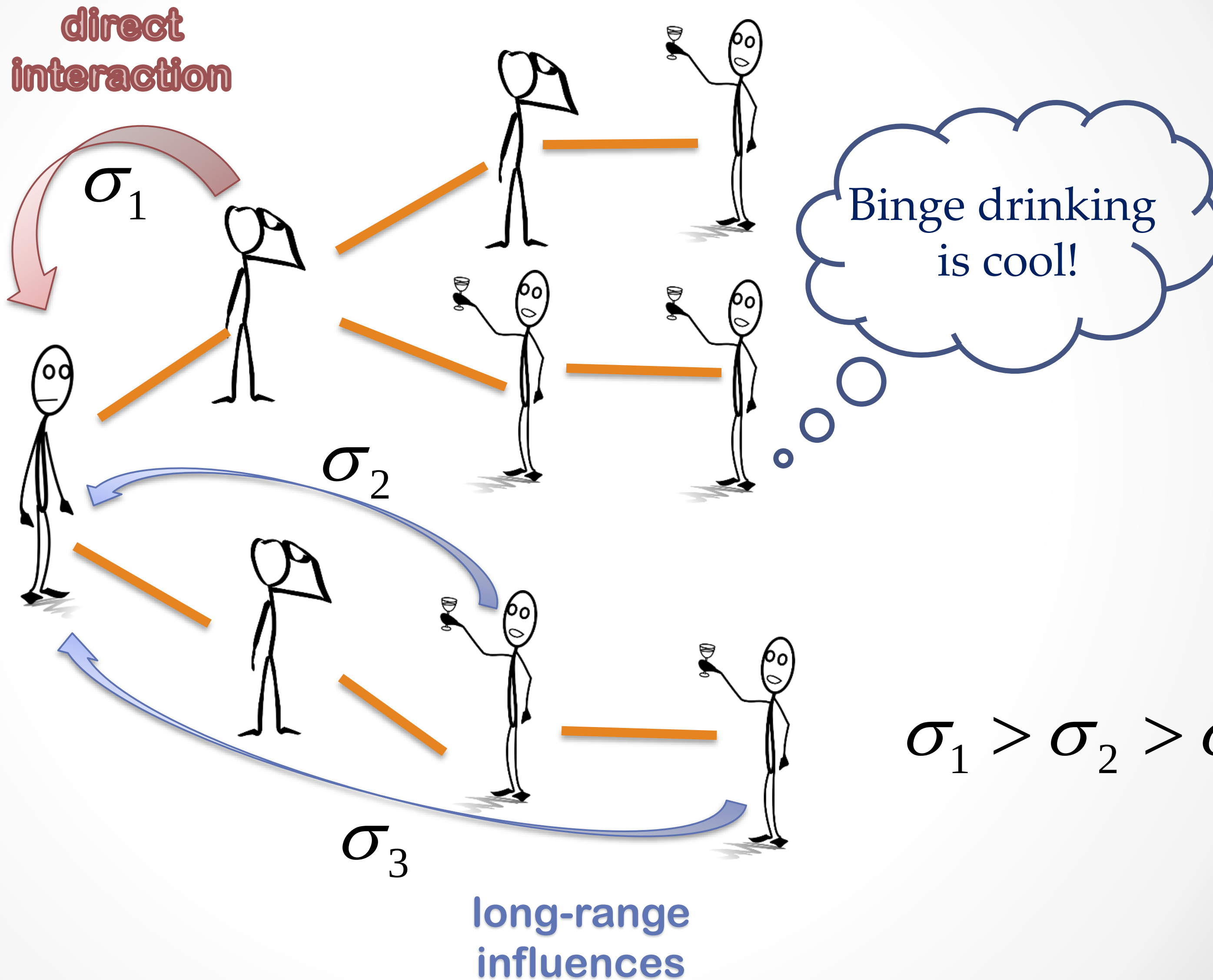
The Network Paradigm

"In a network dynamical processes, at a given time step, "information" is transmitted to nearest-neighbours ONLY"



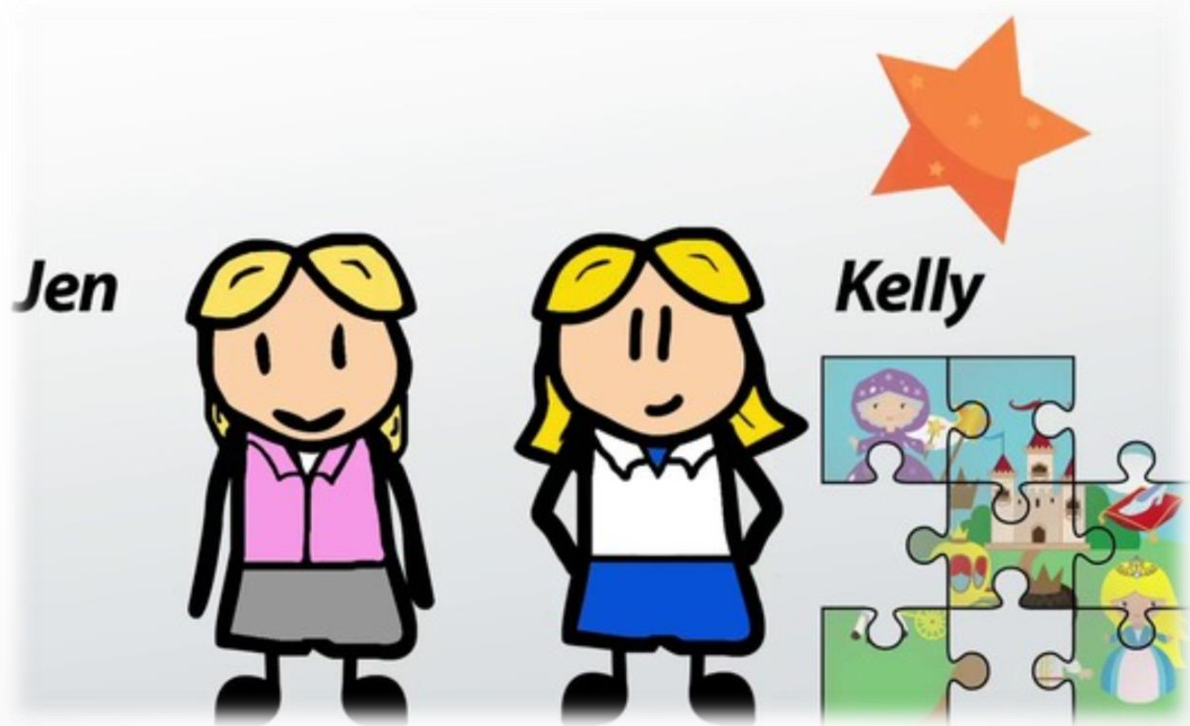
What IF…?



$$\sigma_1 > \sigma_2 > \sigma_3$$

Indirect peer pressure

“**direct peer pressure** may involve verbal and nonverbal attempts at persuasion,
indirect peer pressure encompasses more subtle forms of influence, such as modeling and vicarious reinforcement”.



Drout & Corsoro, *Soc. Behav. Pers.: Int. J.* **31**, (2003) 535-543.

Examples of indirect social pressure

- *"The indirect social pressure: most families install this type of windows as a symbol of their social condition; Installing thermal insulated windows turned into a "fashion"."*

Iancu, Bogdan, Martor 16 (2011) 19-33.

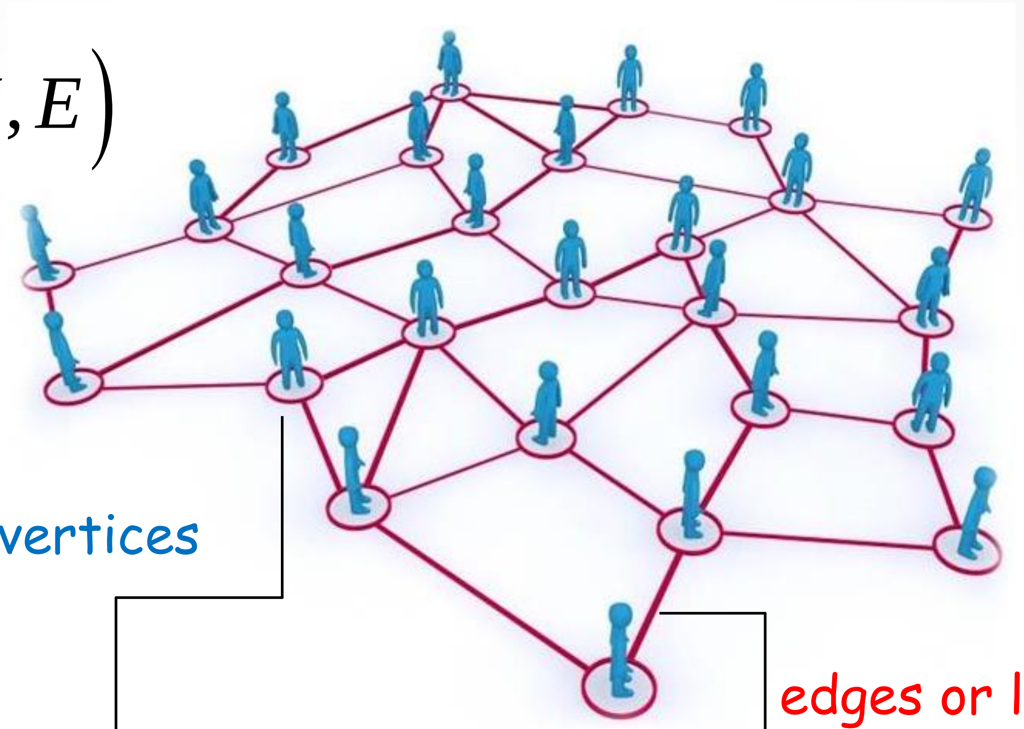
- *Creating a feeling of "if it is OK for them, must be OK for me too"*

- *Use of (social) media:*

Mathematical framework

The mathematical framework

$$G = (V, E)$$



nodes or vertices

edges or links

$$V = \{v_1, \cdots, v_n\}$$

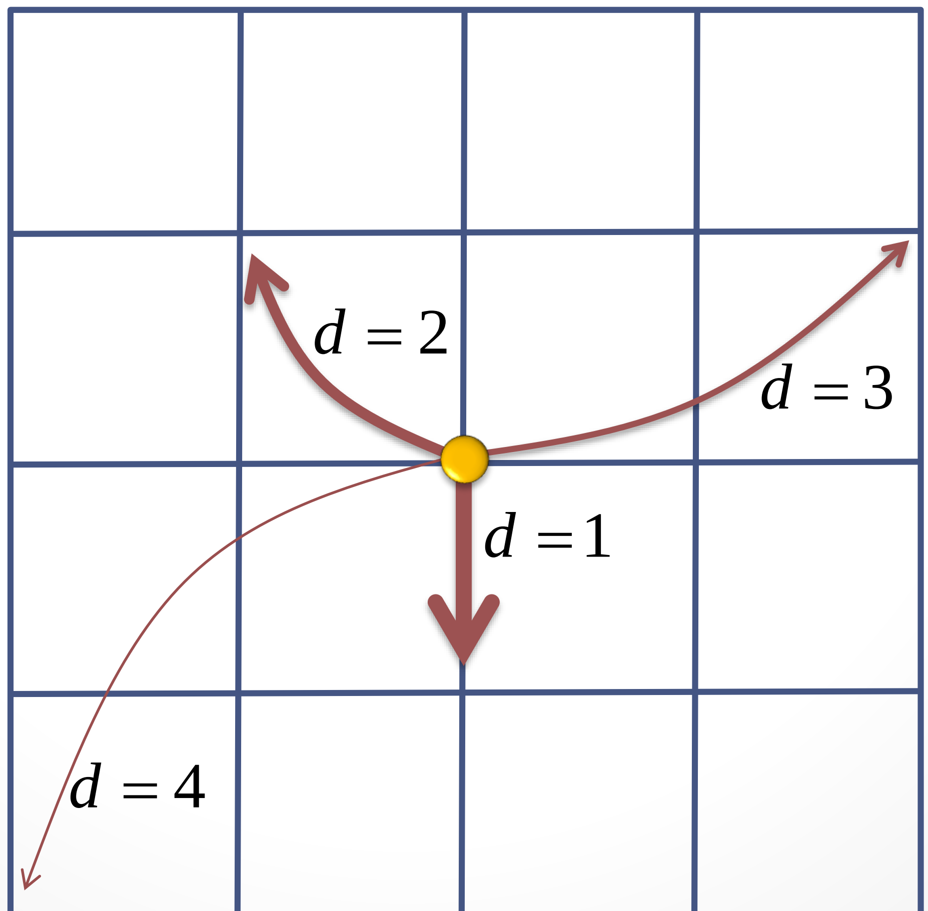
$$E \subseteq V \times V$$

- We consider here simple, finite or locally-finite infinite graphs.

Estrada & Knight: *A First Course on Network Theory*, Oxford Univ. Press, 2015

The mathematical framework

Let us consider that the "information" at a given node can hop not only to its nearest neighbours but to any other node of the network with a probability that decays with the shortest path distance from its current position.



The mathematical framework

Let d be the shortest path distance on $G=(V,E)$.

Let $\delta_d(v)$ be the *d*-path degree of the vertex *v*, i.e.,

$$\delta_d(v) := \#\{w \in V : d(v,w) = d\}.$$

Since G is locally finite, $\delta_d(v)$ is finite for every $v \in V$.

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

The mathematical framework

Let $C(V)$ be the set of all complex-valued functions on V.

Let $\ell^2(V)$ be the Hilbert space of square-summable functions on V with inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v)\bar{g}(v), \qquad f, g \in \ell^2(V)$$

Definition: *The d-path Laplacian operator on graphs is the following mapping of* $C(V)$ *into itself:*

$$(L_d f)(v) := \sum_{w \in V : d(v,w) = d} \left(f(v) - f(w) \right), \qquad f \in C(V).$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

The mathematical framework

Let $e_v \ , v \in V,$ be a standard orthonormal basis in $\ell^2(V)$, where

$$e_v(w) = \begin{cases} 1 & \text{if} \quad w = v \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$(L_d e_v)(w) = \begin{cases} \delta_d(v) & \text{if} \quad w = v \\ -1 & \text{if} \ \ d(v,w) = d \\ 0 & \text{otherwise.} \end{cases}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

Transformed Laplacian operators

Let:

$$\tilde{L} := \sum_{d=1}^{\infty} c_d L_d, \quad c_d \in \mathbb{R}^+$$

If all L_d are bounded and

$$\sum_{d=1}^{\infty} |c_d| \|L_d\| < \infty,$$

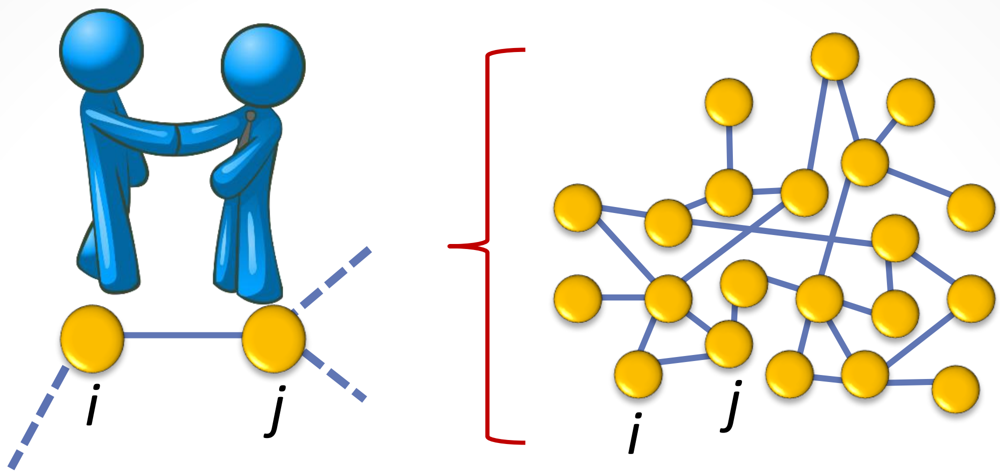
Then, $\tilde{L}$ is:

- *a bounded operator on* $\ell^2(V)$ *,*
- *self-adjoint and,*
- *non-negative.*

Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.


Consensus

Consensus on networks



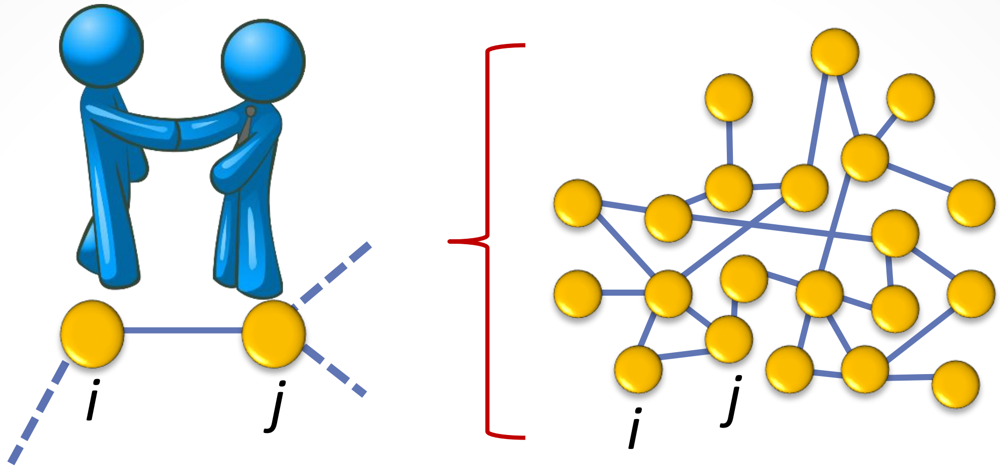
state of j at time t — state of i at time t

$$\frac{du_i}{dt} = \dot{u}_i(t) = \sum_{(i,j)\in E} \left[u_j(t) - u_i(t)\right], \qquad i = 1, \cdots, n$$

$$u_i(0) = z_i, \qquad z_i \in \mathbb{R}$$

Estrada, in: *Evolutionary Equations with Applications in Natural Sciences*. Springer International Publishing, 2015. 93-131.

Consensus on networks



In matrix-vector form:

$$\dot{\vec{u}}(t) = -L(G)\vec{u}(t), \qquad \vec{u}(0) = \vec{u}_0$$

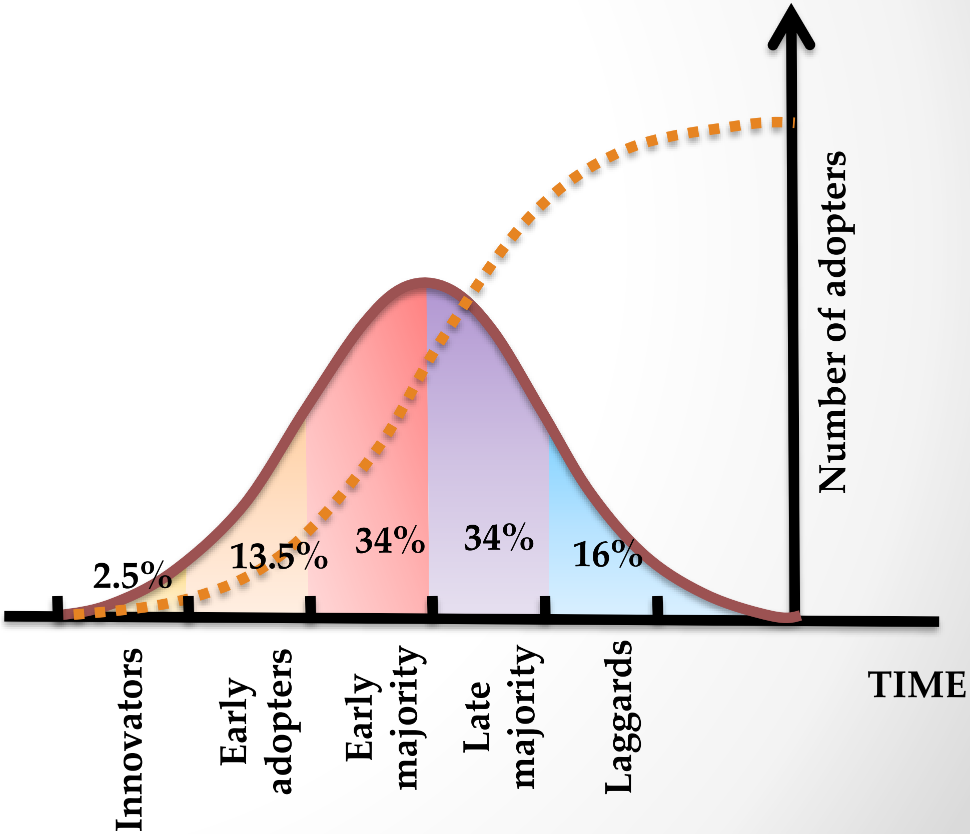
where

$$L_{uv} = \begin{cases} -1 & \text{if } (uv) \in E, \\ k_u & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Consensus on networks

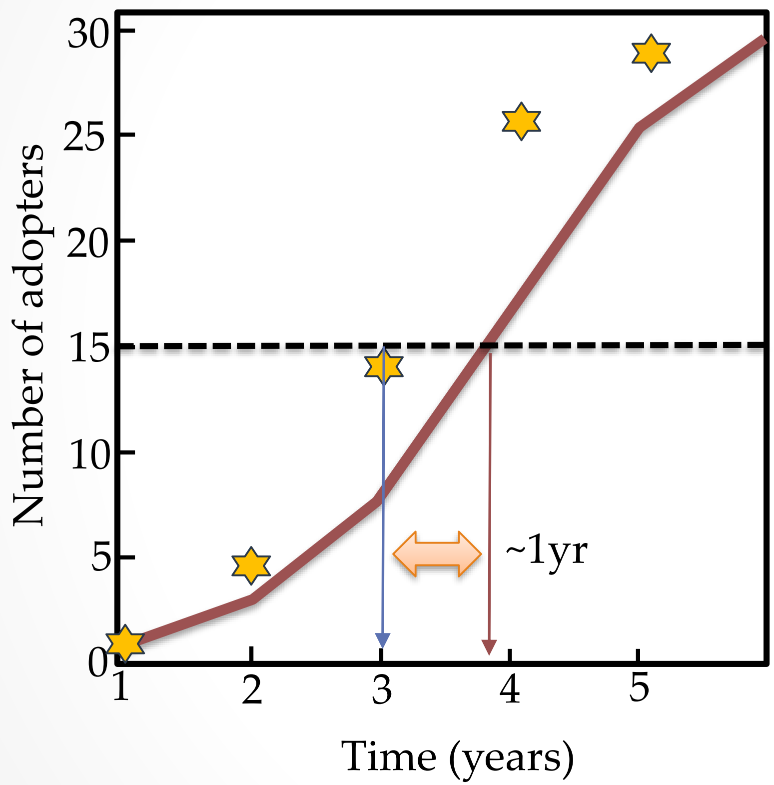
SOCIAL CONSENSUS

DIFFUSION OF INNOVATIONS

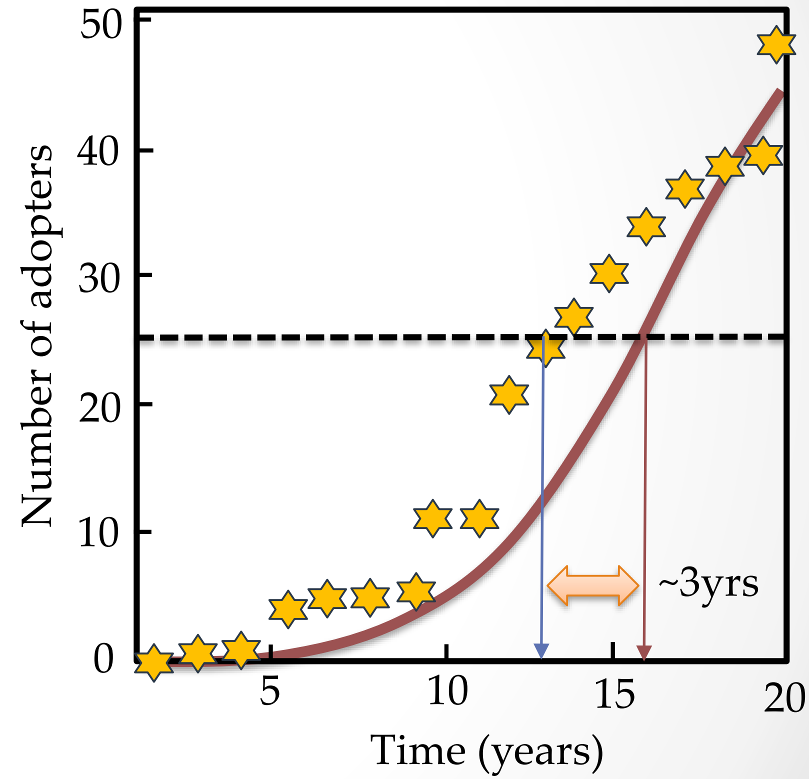


Diffusion of innovations

Diffusion of a mathematical method among high schools



Diffusion of a biotech product among Brazilian farmers



Observation

Normal diffusion

Estrada, Vargas-Estrada, (*Nature*) *Sci*. *Rep*., **3**, 2013.

Generalised consensus on networks

We can now generalise the diffusion dynamics equation to account for such long-range hops:

$$\dot{\vec{u}}(t) = -\tilde{L}\vec{u}(t) \qquad u(0) = u_0$$

$$\tilde{L}_{t=\{s,\lambda\}} = \sum_{d=1}^{\Delta} c_d L_d$$

Mellin transform

$$c_d = d^{-s},$$

$$s \in \mathbb{R}^+$$

Laplace transform

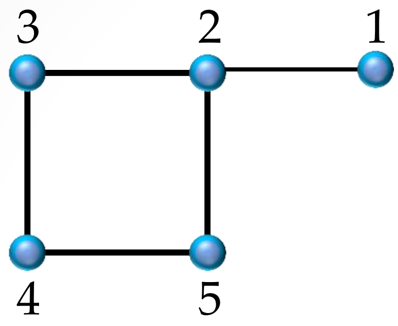
$$c_d = \exp(-\lambda \cdot d),$$

$$\lambda \in \mathbb{R}^+$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

Generalised consensus on networks

Example:



$$\tilde{L}_s = \sum_{d=1}^{\Delta} c_d L_d$$

Mellin transformed Laplacian

$$\tilde{L}_s = \begin{bmatrix} 1+2\cdot 2^{-s}+3^{-s} & -1 & -2^{-s} & -3^{-s} & -2^{-s} \\ -1 & 3+2^{-s} & -1 & -2^{-s} & -1 \\ -2^{-s} & -1 & 2+2\cdot 2^{-s} & -1 & -2^{-s} \\ -3^{-s} & -2^{-s} & -1 & 2+2^{-s}+3^{-s} & -1 \\ -2^{-s} & -1 & -2^{-s} & -1 & 2+2\cdot 2^{-s} \end{bmatrix}$$

Estrada, *Lin. Algebra Appl.*, **436**, 3373, 2012.

Generalised consensus on networks

Example:

$$\tilde{L}_s = \begin{bmatrix} 1+2\cdot 2^{-s}+3^{-s} & -1 & -2^{-s} & -3^{-s} & -2^{-s} \\ -1 & 3+2^{-s} & -1 & -2^{-s} & -1 \\ -2^{-s} & -1 & 2+2\cdot 2^{-s} & -1 & -2^{-s} \\ -3^{-s} & -2^{-s} & -1 & 2+2^{-s}+3^{-s} & -1 \\ -2^{-s} & -1 & -2^{-s} & -1 & 2+2\cdot 2^{-s} \end{bmatrix}$$

$s \to \infty$

$s \to 0$

$$L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

$$L(K_n) = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

How fast
can it go?

Recall that…

normal diffusion

$$MSD \sim t$$

$$\left(u(t)\right)_x = \frac{1}{2\sqrt{\pi \alpha t}} \exp\left(-\frac{x^2}{4at}\right)$$

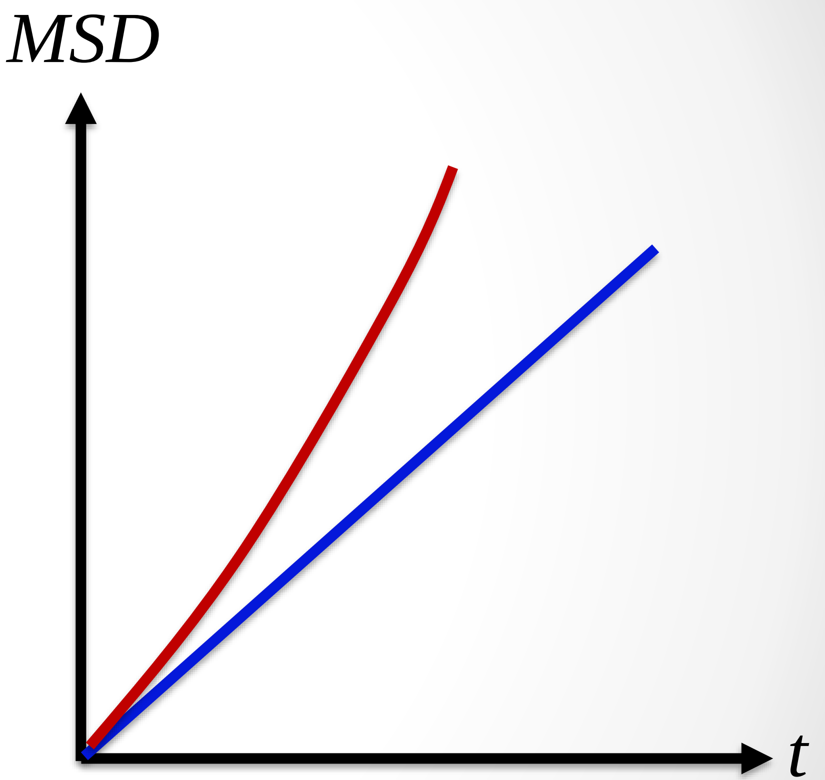
$$FWHM(t) \sim t^{1/2}$$

superdiffusion

$$MSD \sim t^{\alpha}, \alpha > 1$$

$$\left(u(t)\right)_x = t^{-1/\alpha} f\left(t^{-1/\alpha} x; \alpha, 0, \gamma, 0\right)$$

$$FWHM(t) \sim t^{\alpha}, \quad \alpha > 1/2$$



GDE in one-dimension

Theorem: *Let* P_∞ *be the infinite path graph, let* $\lambda > 0$ *and* $\tilde{L}_\lambda$ *be the Laplace-transformed d-path Laplacian with parameter* λ. *Moreover, let* $u(t)$ *be the solution of the generalised diffusion equation with* $L = L_d$ *or* $L = \tilde{L}_\lambda$. *Then, as* $t \to \infty$

$$(u(t))_x = t^{-1/2} \frac{1}{2\sqrt{\pi \alpha}} \exp\left(-\frac{x^2}{4at}\right) + o\left(t^{-1/2}\right)$$

uniformly in $x \in \mathbb{Z}$ *where*

$$a = d^2 \qquad \textit{for} \quad L = L_d$$

$$a = \frac{e^\lambda \left(e^\lambda + 1\right)}{\left(e^\lambda - 1\right)^3} = \frac{\coth\left(\frac{\lambda}{2}\right)}{2(\cosh \lambda - 1)} \qquad \textit{for} \quad L = \tilde{L}_\lambda .$$

Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

Resume

Laplace-transform

$$\left(u(t)\right)_x = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4at}\right)$$

$$FWHM(t) = 2\sqrt{\ln(2)at}$$

$$FWHM(t) \sim t^{1/2}$$

NOTHING NEW

NORMAL DIFFUSION

GDE in one-dimension

Theorem: *Let* P_∞ *be the infinite path graph, let* $s>1$ *and* $\tilde{L}_s$ *be the Mellin-transformed k-path Laplacian with parameter* s. *Moreover, let* $u(t)$ *be the solution of the generalised diffusion equation with* $L=\tilde{L}_s$. *Then*

$$(u(t))_x = t^{-1/\alpha} f\left(t^{-1/\alpha}x;\alpha,0,\gamma,0\right)+o\left(t^{-1/\alpha}\right) \quad as \quad t\to\infty$$

uniformly in $x\in\mathbb{Z}$ *where*

$$\alpha = s-1, \qquad \gamma = \left(-\frac{\pi}{\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}\right)^{\frac{1}{s-1}} \qquad if \quad 1<s<3$$

$$\alpha = 2, \qquad \gamma = \sqrt{\zeta(s-2)} \qquad if \quad s>3.$$

Estrada et al., *Lin. Algebra Appl*., **523**, 307-334, 2017.

Resume

Mellin-transform $1 < s < 3$

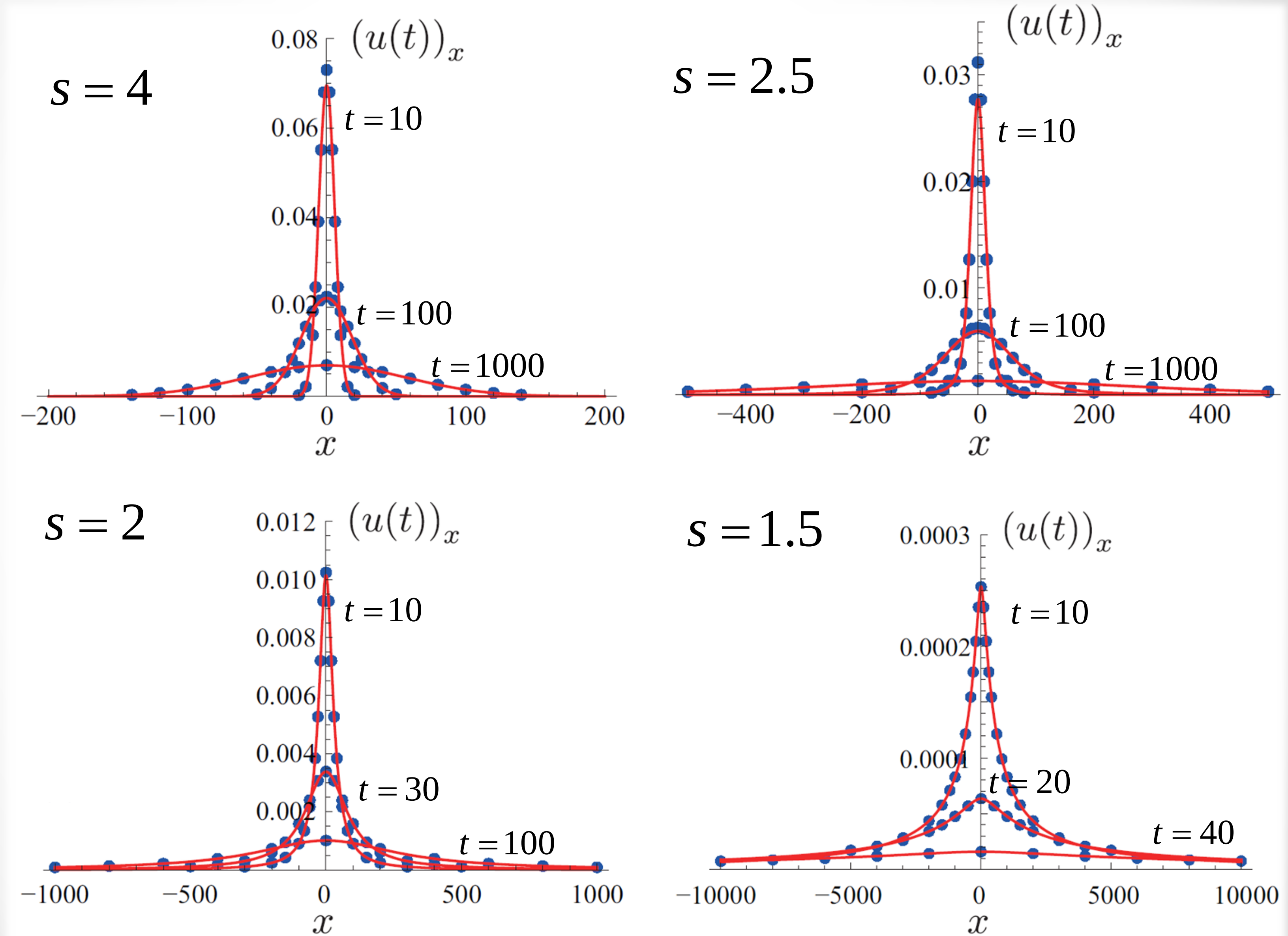
$$(u(t))_x = t^{-1/\alpha} f\left(t^{-1/\alpha} x; \alpha, 0, \gamma, 0\right) + o\left(t^{-1/\alpha}\right)$$

$$FWHM(t) \sim 2\xi_0 t^{1/(s-1)}$$

SUPERDIFFUSION (!)

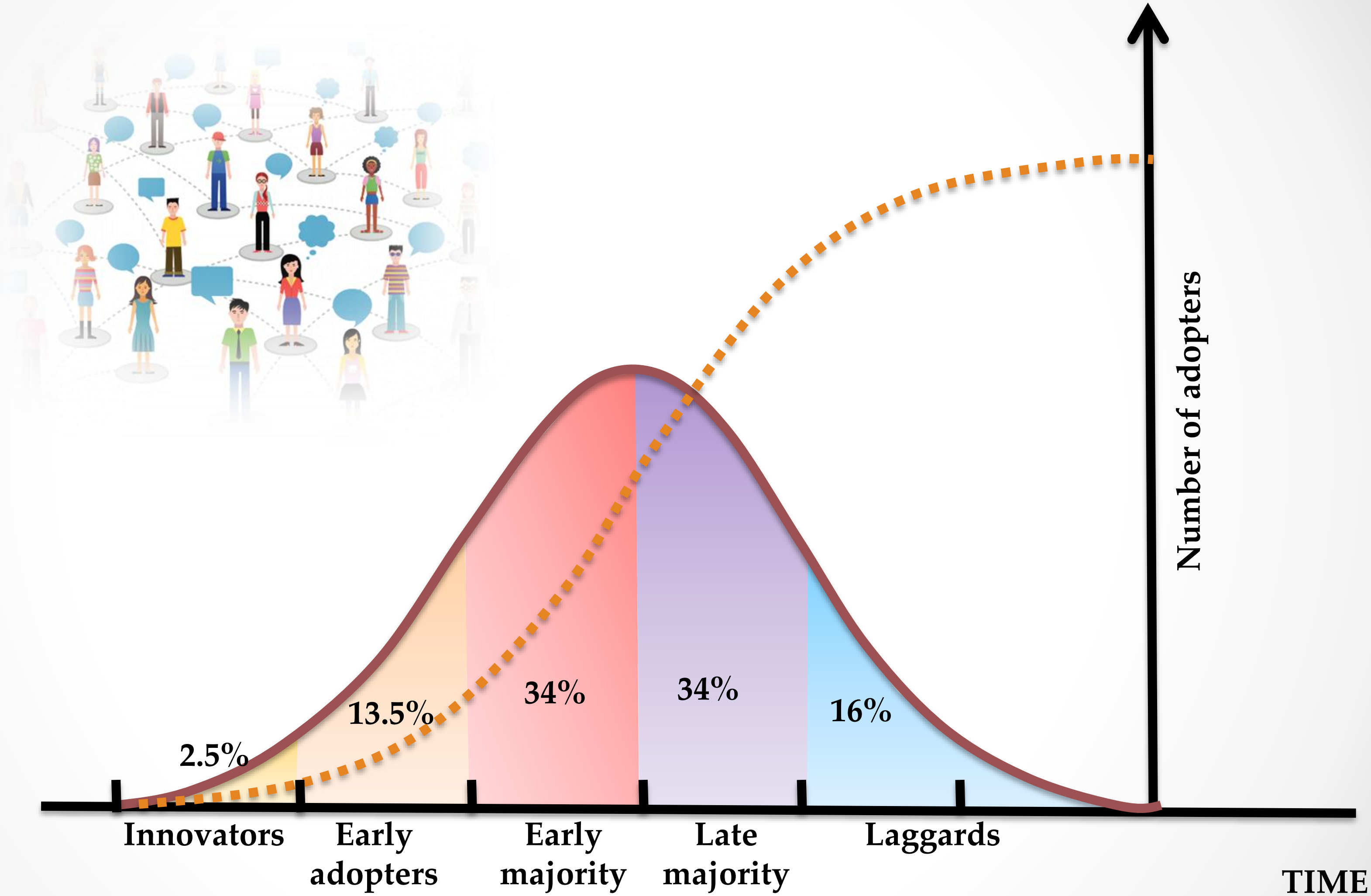
Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

GDE in one-dimension



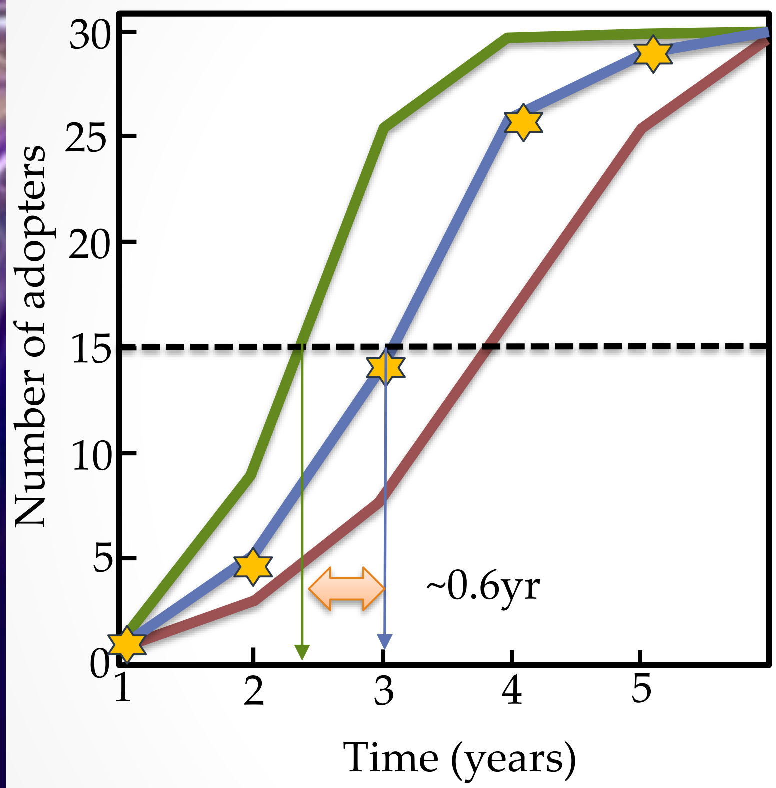
Estrada et al., *Lin. Algebra Appl.*, **523**, 307-334, 2017.

Back to diffusion of innovations

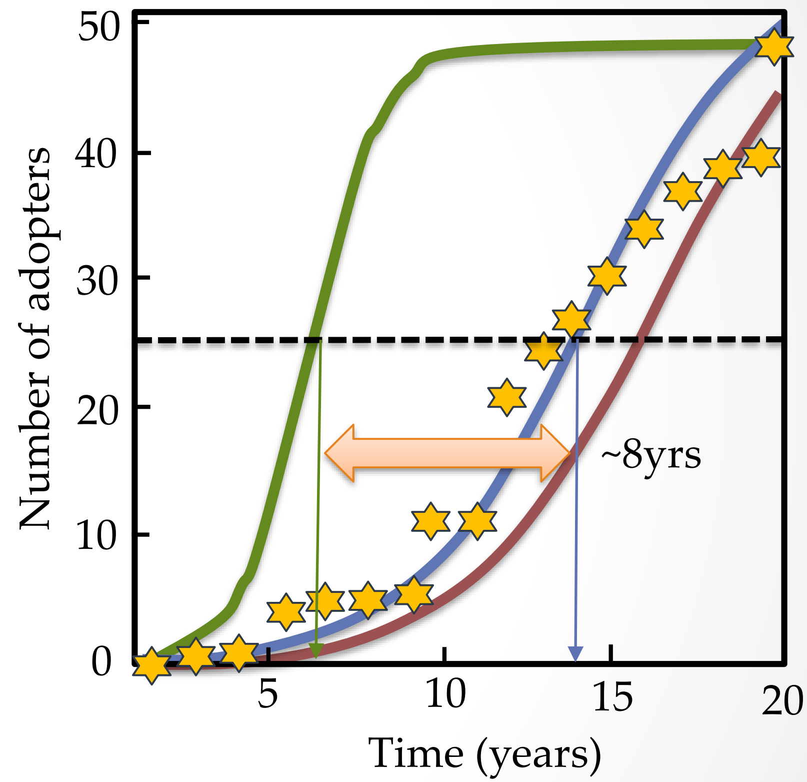


GDE and diffusion of innovations

Diffusion of a mathematical method among high schools



Diffusion of a biotech product among Brazilian farmers



Observation

No indirect peers pressure

Moderate indirect peers pressure

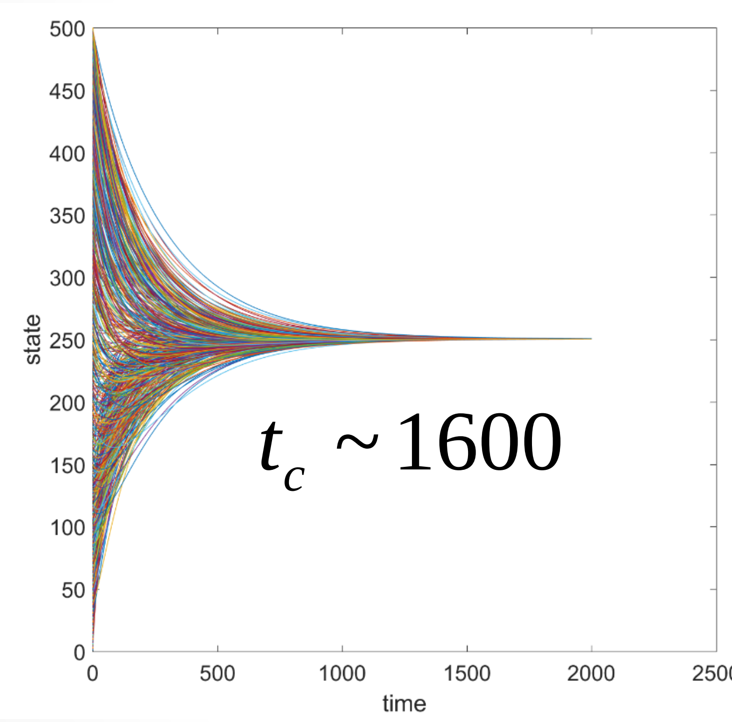
High indirect peers pressure

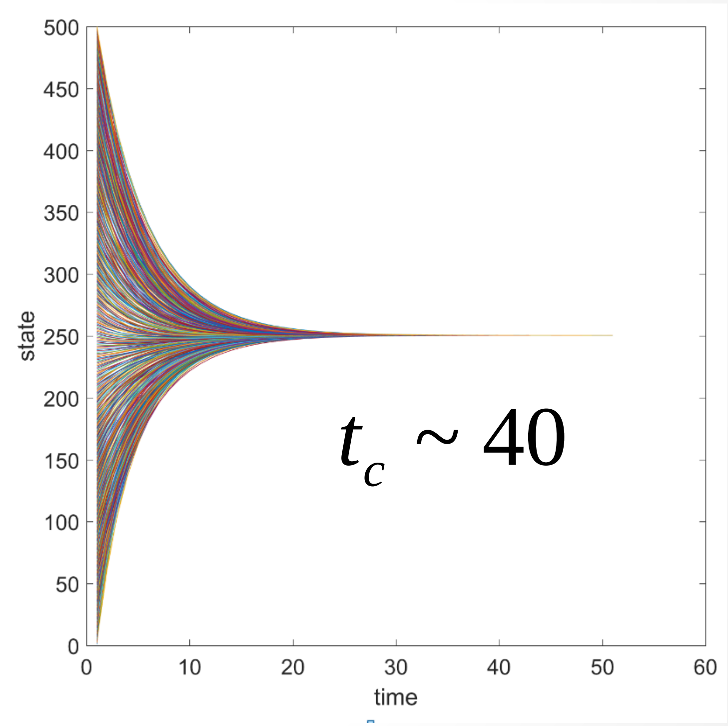
Estrada, Vargas-Estrada, *(Nature) Sci. Rep.*, **3**, 2013.

How much acceleration of consensus?

Barabási-Albert Random Graph

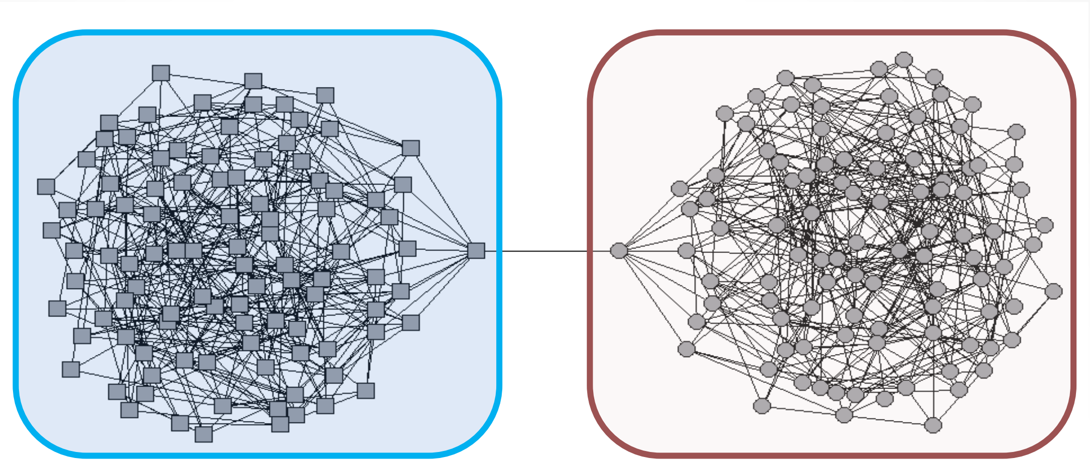
$$c_k = k^{-1}$$



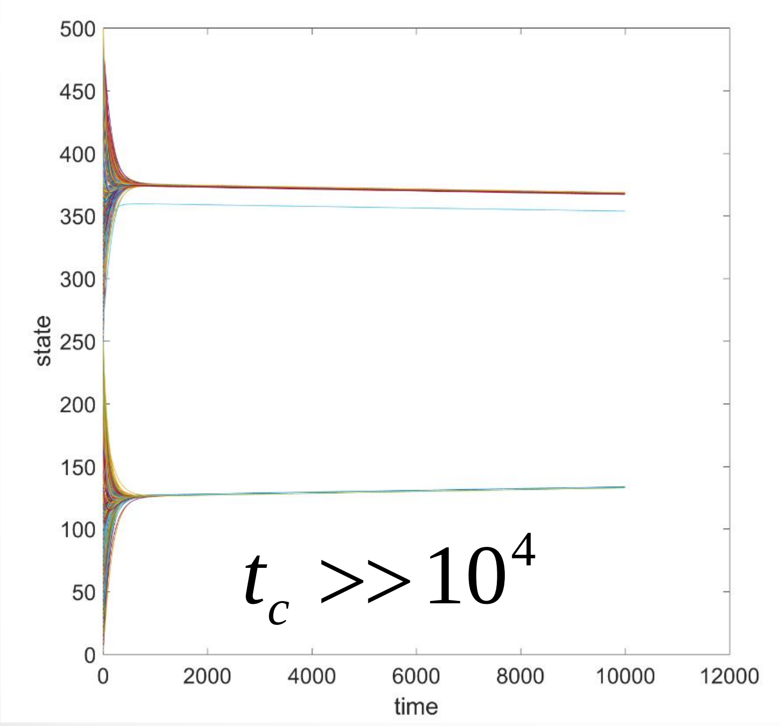


$$G(500,6)$$

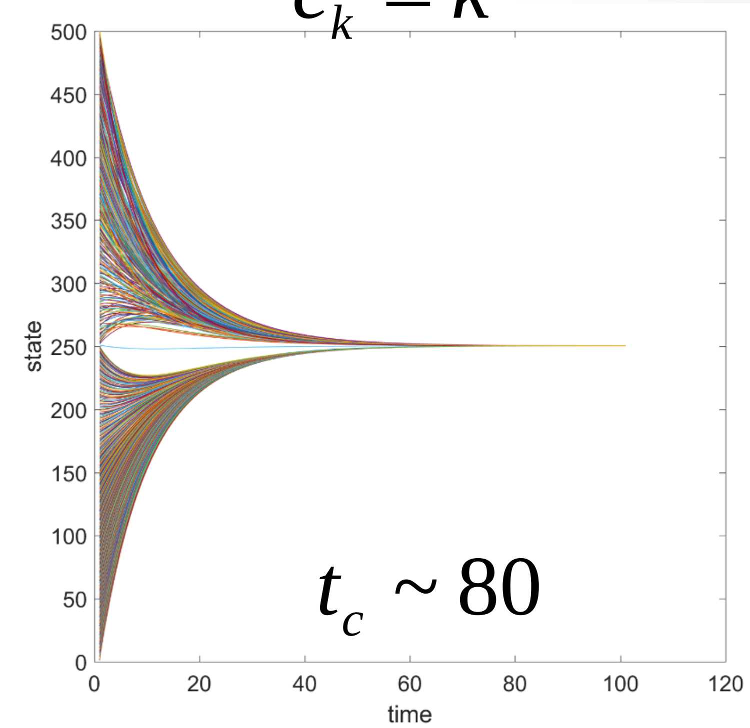
How much acceleration of consensus?



$$c_k = k^{-1}$$



$$t_c \gg 10^4$$

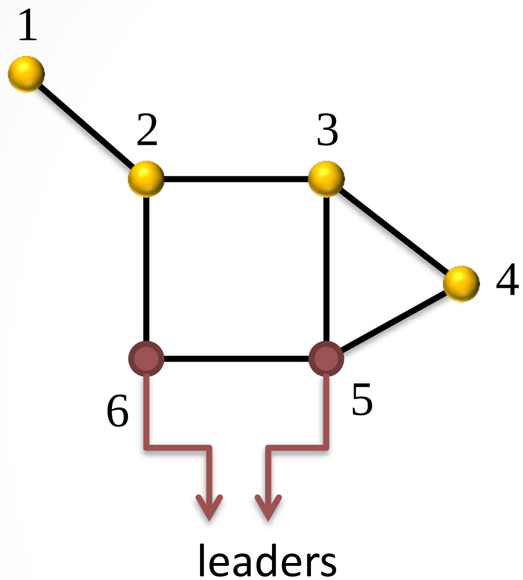


$$t_c \sim 80$$

Leader-Followers Consensus

Let us consider the partition of the network into n_l leaders and n-n_l followers.

Example:



followers

$$\mathbf{L}=\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

leaders

$$\mathbf{L}_l=\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \qquad \mathbf{L}_f=\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad \mathbf{L}_{fl}=\begin{pmatrix} 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ -1 & 0 \end{pmatrix}$$

Leaders-followers consensus

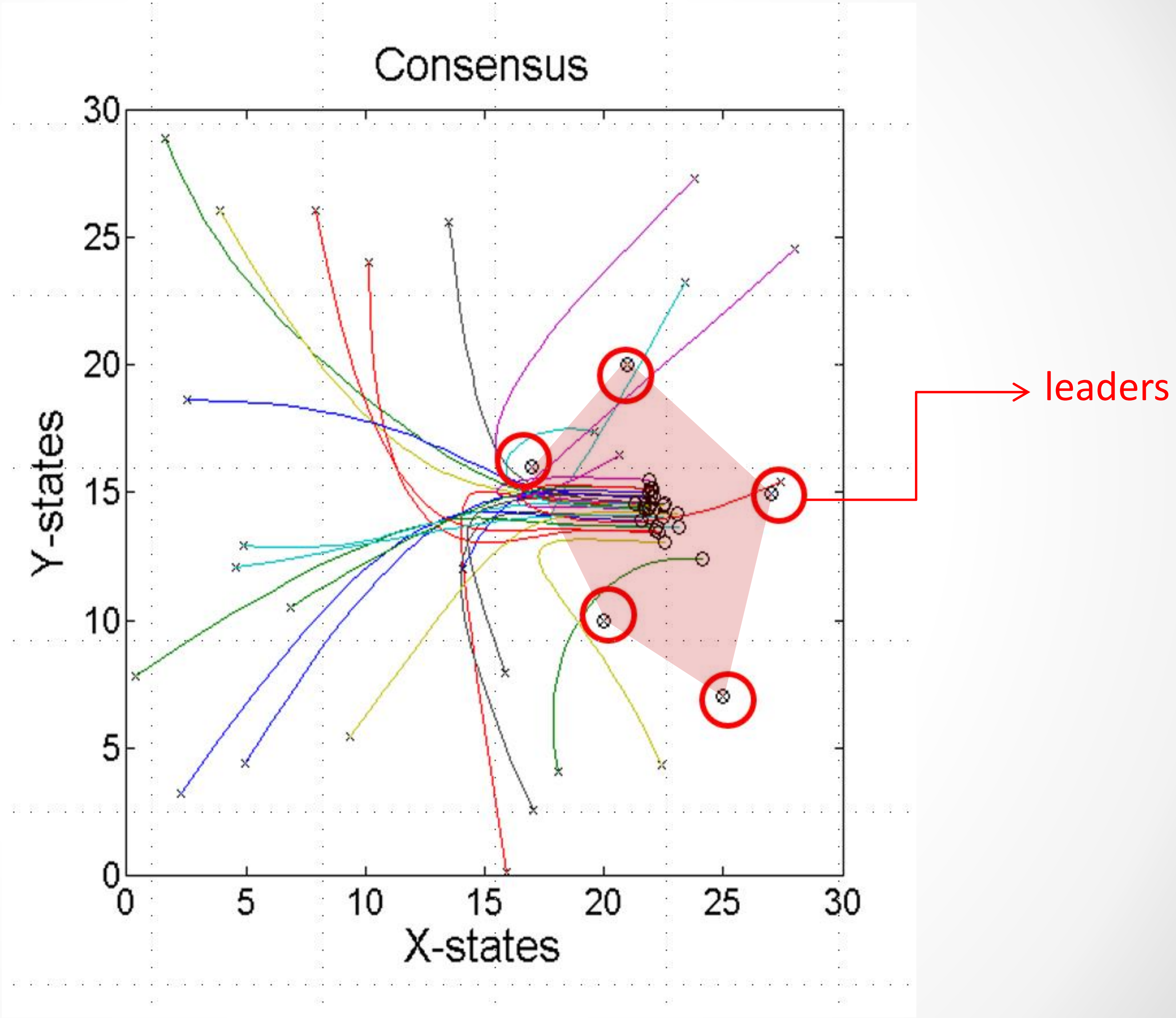
The consensus dynamics of a leaders-followers system is described by:

$$\begin{bmatrix} \dot{\mathbf{u}}_f \\ \dot{\mathbf{u}}_l \end{bmatrix} = -\begin{bmatrix} \mathbf{L}_f & \mathbf{L}_{fl} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_f \\ \mathbf{u}_l \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}$$

$$\dot{\mathbf{u}}_f = -\mathbf{L}_f \mathbf{u}_f - \mathbf{L}_{fl} \mathbf{u}_l$$

Mesbahi & Egerstedt., *Graph Theory Methods in Multiagent Networks*, Princeton, 2010 .

Leaders-followers consensus



Who is the best Leader?

Leaders selection

Let us consider that the **best leader** is the one that leads to the fastest consensus of its followers.

Theorem: *The time of consensus averaged over all the nodes in the network is bounded as follows:*

$$\langle t_c \rangle \geq \frac{1}{n\mu_2} \sum_{p=1}^{n} \ln \left| \frac{\vec{\psi}_2 (p) \left(\vec{\psi}_2 \cdot \vec{u}_0 \right)}{\delta} \right| .$$

Estrada, Sheerin, *Physica D, Nonlinearity* **323-324**, 2016, 20-26.

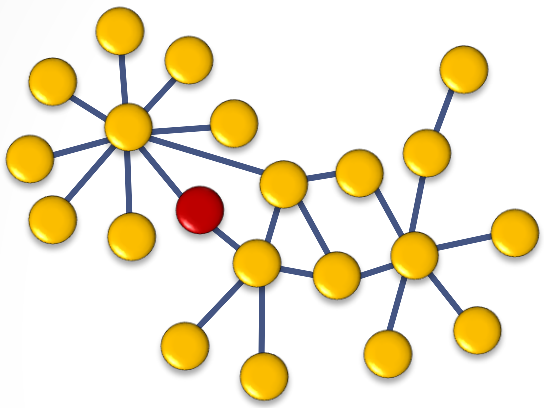
Is the *most central* agent the best leader?

Agents centrality

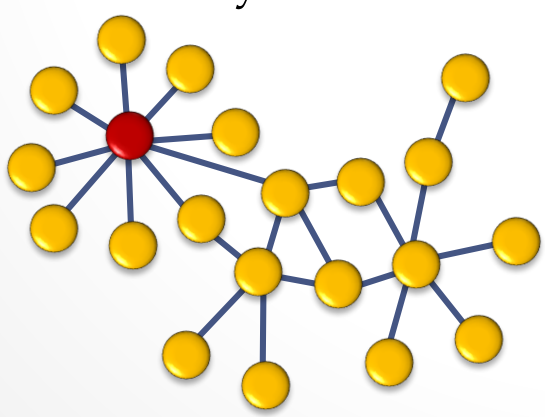
BC: betweenness centrality
CC: closeness centrality
DC: degree centrality
EC: eigenvector centrality
SC: subgraph centrality

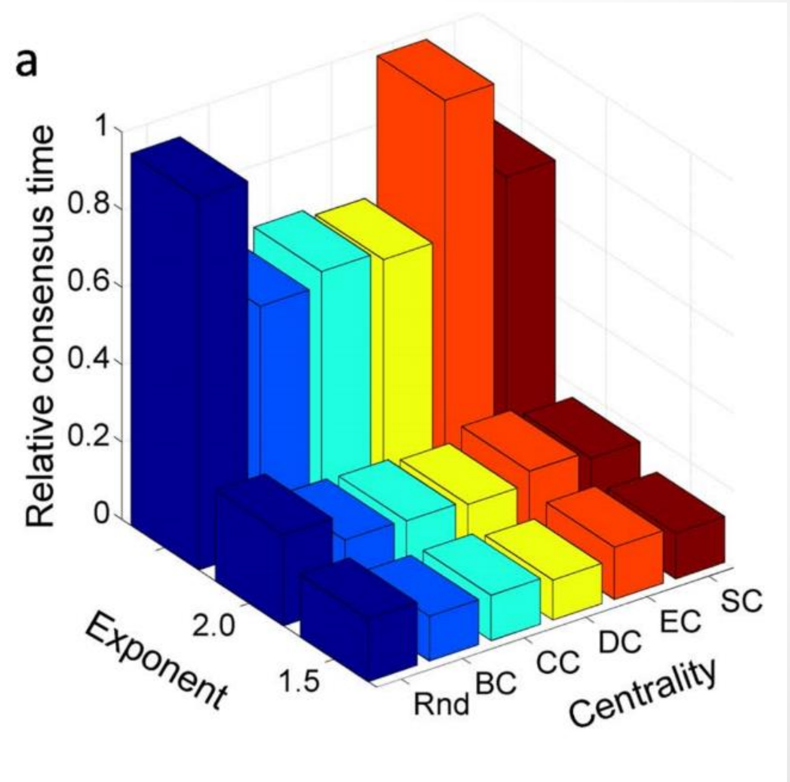
Leaders selection

Random selection



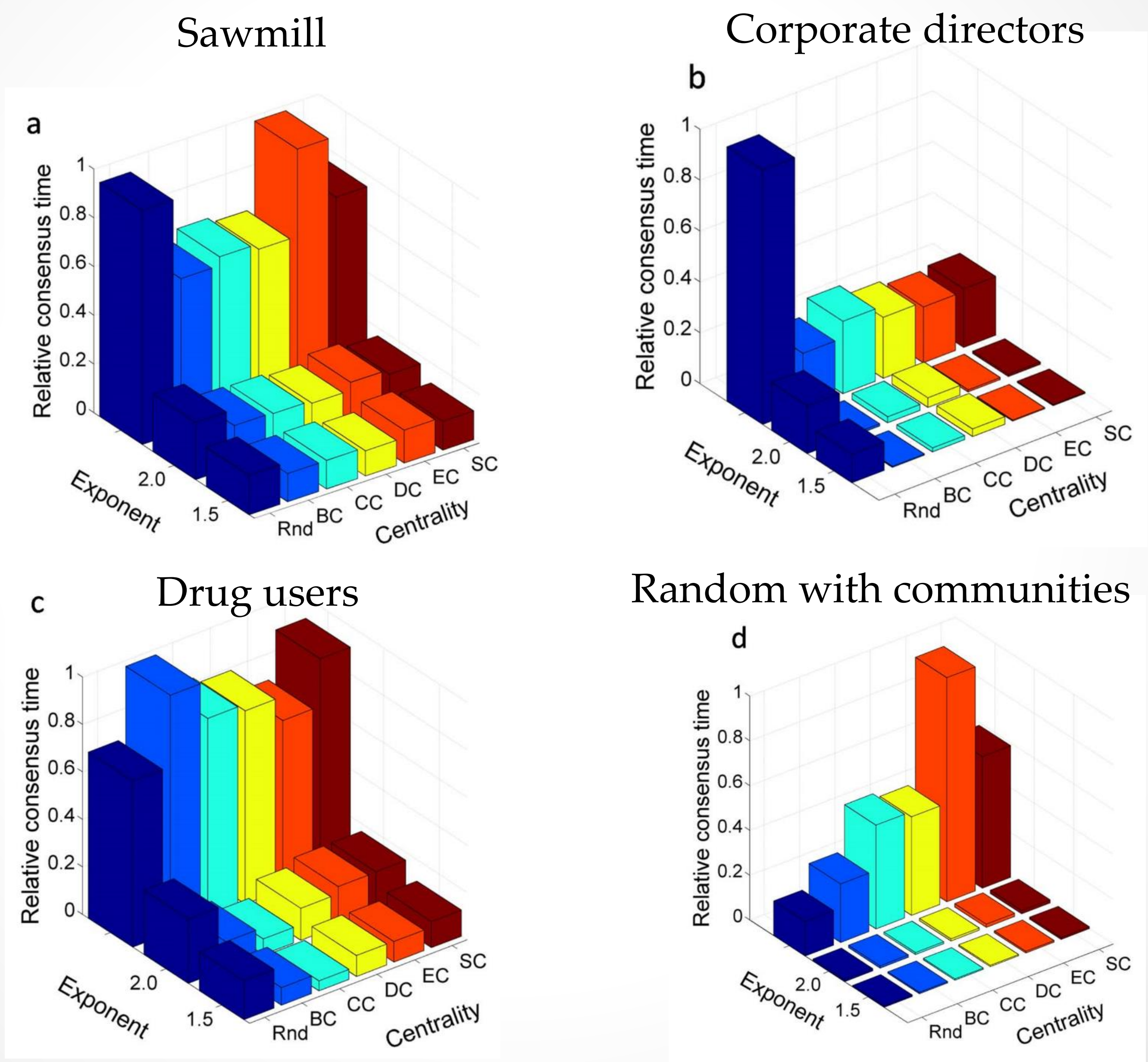
Centrality-based





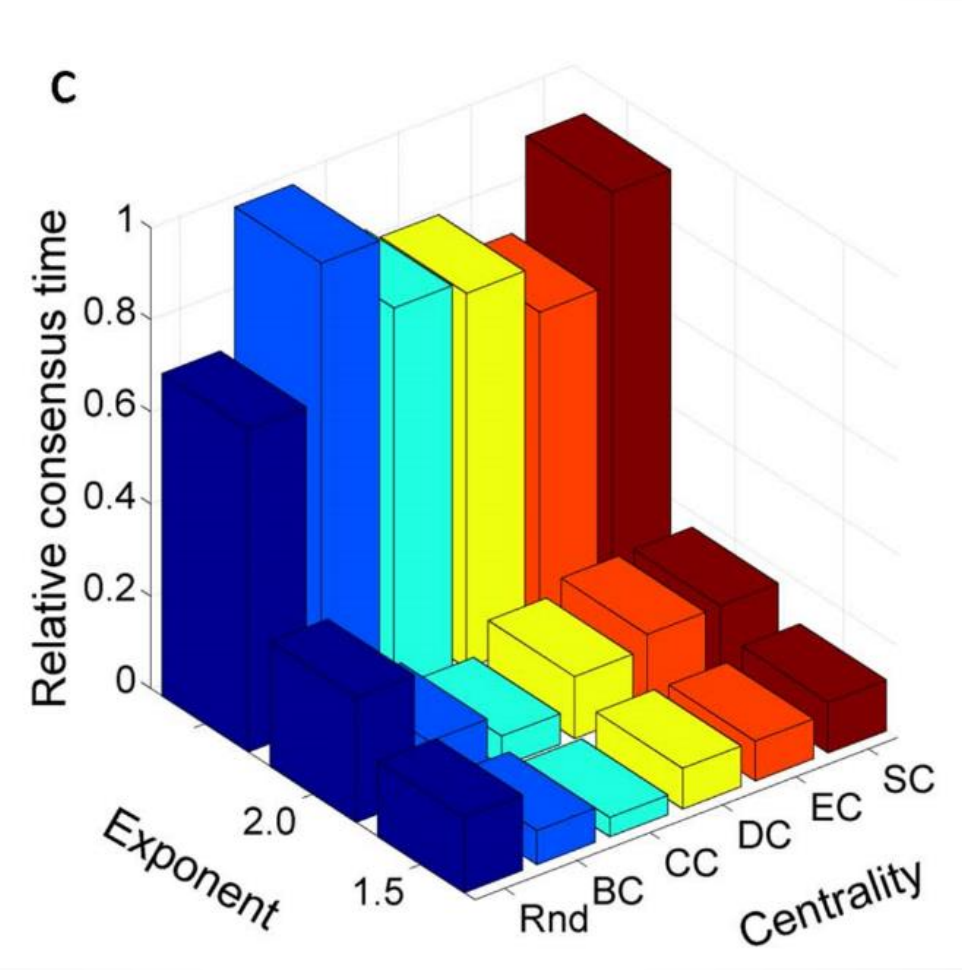
Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

Leaders selection

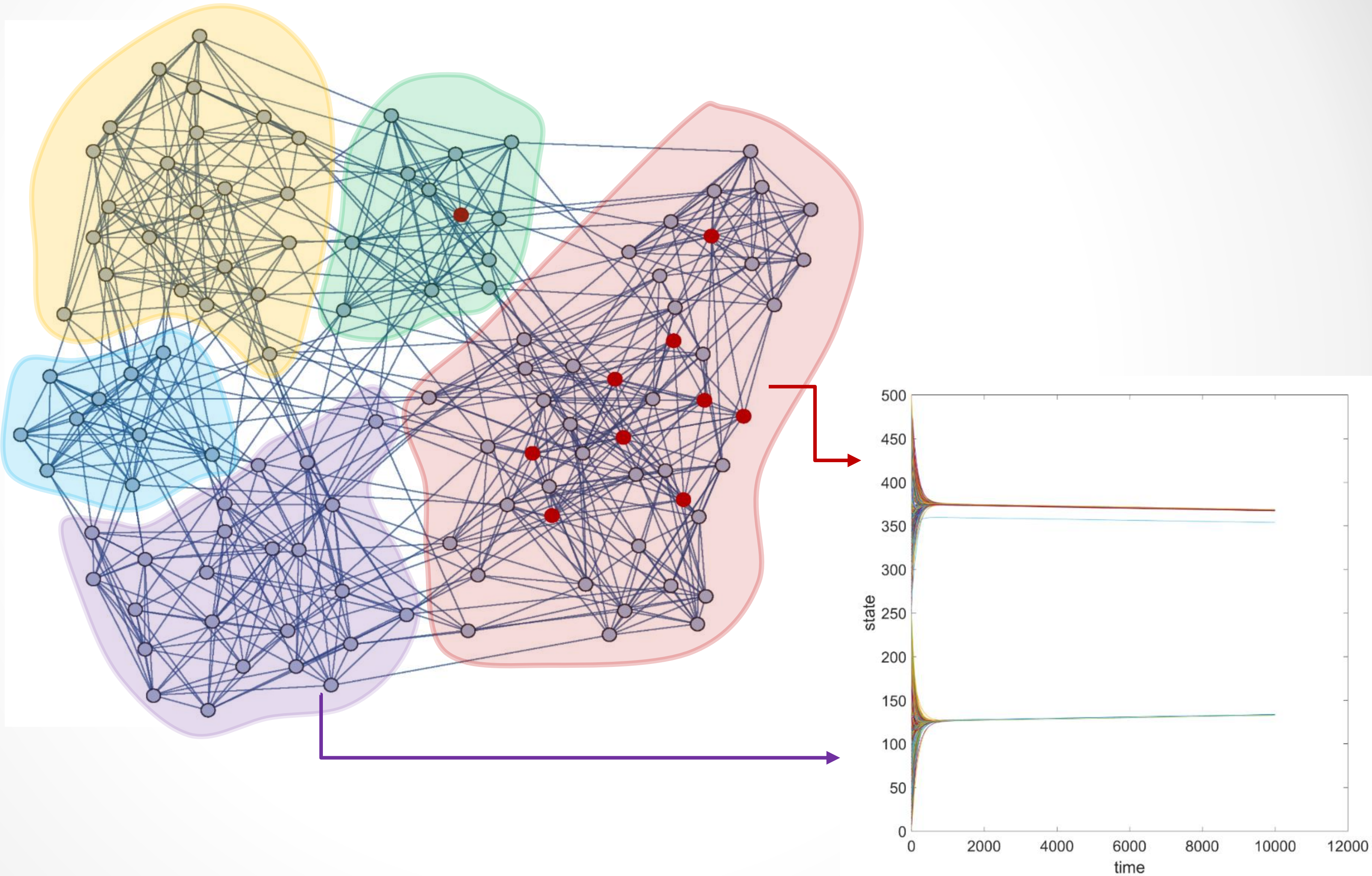


When random is better…

Injecting drug users



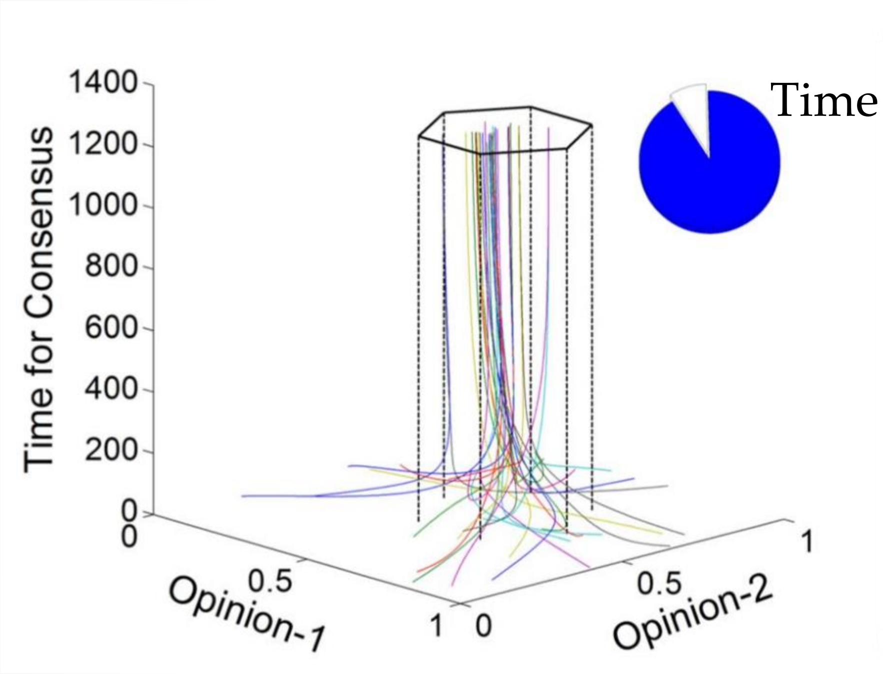
Or, the need of local leaders

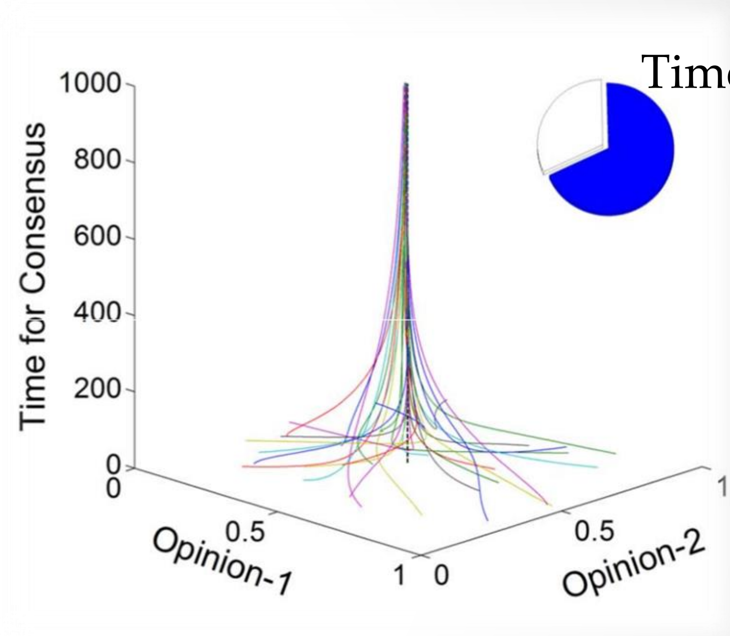


Leaders selection

LEADERS' COHESIVENESS

Leaders selection





LEADERS' COHESIVENESS

Estrada, Vargas-Estrada, *Sci*. *Rep*., **3**, 2013.

Sync under long-range influences

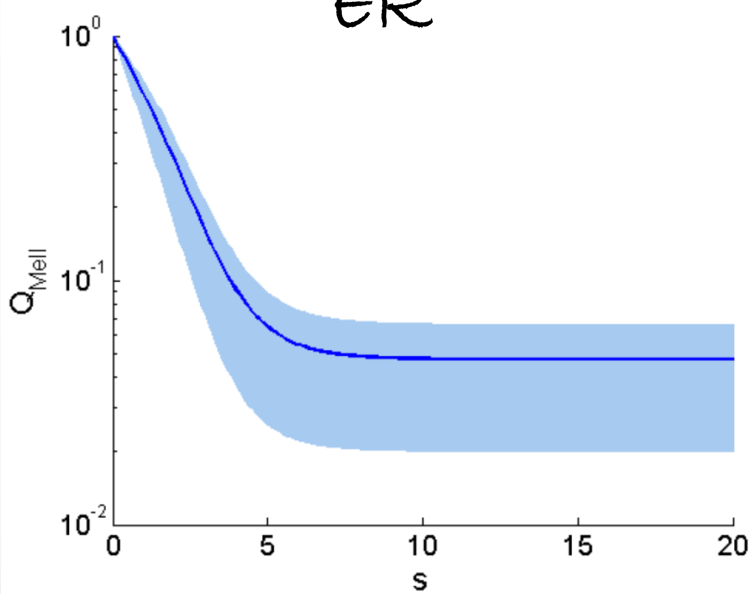
Generalised sync model

$$\dot{\vec{\theta}} = \vec{\omega} - \sigma \left(\sum_{d=1}^{d_{max}} d^{-s} \nabla_d \cdot \sin\left(\nabla_d^T \vec{\theta} \right) \right),$$

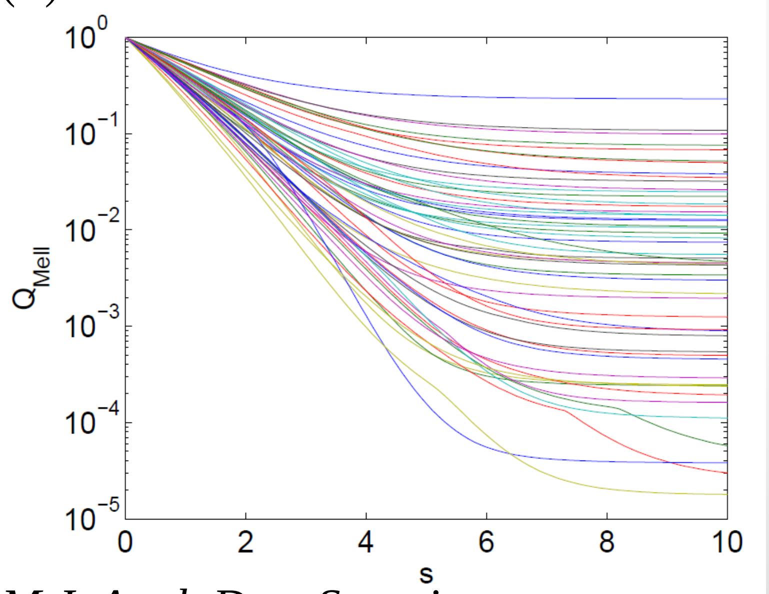
$$\dot{\vec{\theta}} = \vec{\omega} - \sigma \left(\nabla \cdot \sin\left(\nabla^T \vec{\theta} \right) \right) - \sigma \left(\sum_{d=2}^{d_{max}} e^{-\lambda d} \nabla_d \cdot \sin\left(\nabla_d^T \vec{\theta} \right) \right).$$

$$Q := \frac{\lambda_2(s)}{\lambda_n(s)}$$

ER



Real-world



Estrada, Gambuzza, Frasca, *SIAM J. Appl. Dyn. Syst.*, in press.
arXiv: 1704.01349 (**2017**).

Conclusions

- A generalisation of the *consensus dynamics model* is done to include *long-range influences* (LRI).
- LRI influences significantly the *rate of consensus* in networks.
- LRI may give rise to *superdiffusive behaviour* on networks under certain conditions.
- LRI influences significantly on *leaders selection* in networks.
- The LRI scheme has been extended to other dynamical processes on networks, such as (i) *synchronisation*; (ii) *epidemics* on spatial networks; (iii) *random walks* (iv) *replicator-mutator* dynamics; (v) *nonlinear diffusion*.

Thank you!
Merci!